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EXISTENCE RESULTS FOR A PARABOLIC PROBLEM OF KIRCHHOFF TYPE VIA TOPOLOGICAL DEGREE

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Abstract. In the present paper, we use the topological degree methods to show the existence of weak solutions for a class of nonlinear parabolic problem of Kirchhoff type. Our proof is based on a transformation of this non-local parabolic problem into a new one governed by an operator equation which a sum of a semi-continuous bounded map of type (S_+) with respect to the domain of the second operator which is a linear densely defined maximal monotone operator.

Keywords: Parabolic equation of Kirchhoff type, nonlocal parabolic problem, topological degree, weak solution.
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1 Introduction and motivation

In this paper, we devote our attention to the existence of non-trivial weak solutions for nonlocal (Kirchhoff type) parabolic equations. More precisely, we deal with a class governed by the following parabolic initial-boundary problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - M\Big(\displaystyle\int_\Omega A(x,t,\nabla u)\,dx\Big)\, div\big(a(x,t,\nabla u)\big) = g(x,t) & \text{in} \quad Q, \\ u(x,t) = 0 & \text{on} \quad \partial Q, \\ u(x,0) = u_0(x) & \text{on} \quad \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is an bounded open set with smooth boundary $\partial\Omega$, and p be a real number such that $2 < p < \infty$, $-div\big(a(x,t,\nabla u)\big)$ is a Leray-Lions operator acting from $\mathcal{V} := L^p(0,T;W_0^{1,p}(\Omega))$ to its dual $\mathcal{V}^* := L^{p'}(0,T;W^{-1,p'}(\Omega))$. For $T > 0$, $Q = \Omega \times (0,T)$ denotes the cylinder and $\partial Q = \partial\Omega \times (0,T)$ its boundary. Here, g belongs to $\mathcal{V}^*$.

The problem of the above form (1) is called a non-local problem because due to the presence of the term M, it is no longer a point identity, which causes some mathematical difficulties and also makes the consideration of such a problem particularly interesting. Especially, Problem (1) belongs to parabolic Kirchhoff equation which has seen significant success in the study of population dynamics in recent years Tuan (2020). Moreover, many other phenomena, such as nonlinear elasticity, non stationary fluids, image recovery, and so on, can be modelled using equations like (1), see for example Autuori et al. (2010); Caraballo et al. (2016); Yacini et al. (2021); Temghart et al. (2021).

The nonlocal operator $M\Big(\displaystyle\int_\Omega A(x,t,\nabla u)\,dx\Big)\, div\big(a(x,t,\nabla u)\big)$ generalizes the term $\Big(a + b\displaystyle\int_\Omega |\nabla u|^2 dx\Big)\triangle u$ of the Kirchhoff equation, introduced by Kirchhoff (1883) in their study

of the oscillations of stretched strings and plates, More precisely, Kirchhoff proposed a model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \Big|\frac{\partial u}{\partial x}\Big|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0 \tag{2}$$

where $\rho,\ \rho_0,\ h,\ E,\ L$ are all constants, this equation is an extension of the classical d'Alembert's wave equation for free vibrations of elastic strings by considering the effect of a change in the length of the string during the vibration. However, this type of problem drew the interest of many authors, mostly After the groundbreaking work of Lions (1978), in which a functional analysis technique was proposed to attack it. Since then dozens of articles have fallen, we can cite in particular the works of Chipot & Lovat (1997); Chipot et al. (1992), Corrêa et al. (2004); Corrêa & Figueiredo (2009) and their references.

Now, we present the framework for the results in this paper. Starting with the simplest case $M = 1$, we refer to the pioneering paper by Lions (1969), in which the existence of the solution $u \in \mathcal{V}$ is established for the following nonlinear parabolic Cauchy-Dirichlet problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - div\big(a(x,t,u,\nabla u)\big) = f(x,t), & (x,t) \in Q, \\ u(x,0) = 0, & x \in \Omega, \\ u(x,t) = 0, & (x,t) \in \Gamma, \end{cases} \tag{3}$$

where $-div\big(a(x,t,u,\nabla u)\big)$ is a pseudo-monotone, coercive, uniformly elliptic operator from $\mathcal{V}$ into $\mathcal{V}^*$ and f belongs to $\mathcal{V}^*$. Following that, in Boccardo et al. (1999) were interested in some existence and regularity results for the solutions of (3), depending on the summability of the data f. In this regard, we mention the works Afraites et al. (2022); Asfaw (2017); Hammou & Azroul (2020); Nachaoui et al. (2021, 2016); Rasheed et al. (2021) for more details.

On the other hand, when $A(x,\nabla u) = |\nabla u|^2$ and $f(x,t) = |u|^{q-1}u$, Han et al. (2018) discussed the global existence and finite time blow-up of solutions when the initial energy is subcritical, critical, or supercritical to the parabolic problem with nonlocal diffusion coefficient.

Fu et al. (2016) studied the following parabolic initial boundary problem for nonlocal (Kirchhoff type) parabolic equations involving variable exponent

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Big[a + b\Big(\displaystyle\int_\Omega \frac{|\nabla u|^{p(x,t)}}{p(x,t)}dx\Big)^{r(t)}\Big] div\big(|\nabla u|^{p(x,t)-2}\nabla u\big) + |u|^{p(x,t)-2}u \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = f(x,t,u), \qquad (x,t) \in \Omega\times(0,T), \\ u(x,t) = 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x,t) \in \partial\Omega\times(0,T), \\ u(x,0) = u_0(x), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \in \Omega. \end{cases}$$

Here a, b are given positive constants, $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$. Based on suitable conditions on u_0 and the hypotheses on the functions $r,\ p,\ f$. The authors proved the local existence of weak solutions by means of the Galerkin approximation method.

Motivated by the above papers, we establish the existence of weak solutions to (1) in the space $\mathcal{V}$ by applying the topological degree method. However, the topological degree theory has been used extensively in the study of nonlinear differential equations as a very successful tool, especially those of elliptic type. For more information on the history of this theory and its applications, see for example Abbassi et al. (2020b,a, 2021); Allalou et al. (2021); Adhikari et al. (2021); Berkovits et al. (1992); Berkovits (2007); Cho & Chen (2006).

This document is structured as follows: In the next section, we present the main results of this article. In section 3, we state some necessary preliminary results and we give some related lemmas that will be used in the proof of the main results. Section 4 is devoted to the proof of the main results.

2 The main theorem

In this section, we will give our main theorem. For this, we list our assumptions associated with our problem to show the existence results.

From new on, we always assume that $a(x,t,\xi) : Q \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory vector-valued function, such that $a(x,t,\xi) = \nabla_\xi A(x,t,\xi)$, where $A(x,t,\xi) : Q \times \mathbb{R}^N \longrightarrow \mathbb{R}$. Suppose that a and A satisfy the following hypotheses, for a. e. in $(x,t) \in Q$ and all $\xi, \xi' \in \mathbb{R}^N$, $(\xi \neq \xi')$.

(A_1) $\quad A(x,t,0) = 0,$

(A_2) $\quad a(x,t,\xi) \cdot \xi \geq \alpha |\xi_i|^p,$

(A_3) $\quad |a(x,t,\xi)| \leq \beta \left(d(x,t) + |\xi|^{p-1}\right),$

(A_4) $\quad [a(x,,t,\xi) - a(x,t,\xi')] \cdot (\xi - \xi') > 0,$

where α, β are some positive constants and $k(x,t)$ is a positive function in $L^{p'}(Q)$ (p' is the conjugate exponent of p).
Furthermore,

(M_0) $M : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and non-decreasing function, for which there exist two positive constant m_0 and m_1 such that $m_0 \leq M(s) \leq m_1$ for all $s \in [0, +\infty[$.

Now, we present our main result.

Theorem 1. *Suppose that the hypotheses $(A_1) - (A_4)$ and (M_0) hold. Then for $g \in \mathcal{V}^*$ and $u_0 \in L^2(\Omega)$, problem (1) has a weak solution $u \in D(L)$ in the following sense*

$$-\int_Q uv_t dxdt + \int_0^T M\Big(\int_\Omega A(x,t,\nabla u)dx\Big) \int_\Omega a(x,t,\nabla u)\nabla v dxdt = \int_Q gv dxdt, \tag{4}$$

for all $v \in \mathcal{V}$.

3 Preliminaries

In this part, we present functional framework required to investigate the problem (1), as well as the fundamental definitions and theorems of topological degree theory that are relevant to our goal.

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open set with smooth boundary. Let $p \geq 2$ and $p' = \dfrac{p}{p-1}$. We will denote by $L^p(\Omega)$ the Banach space of all measurable functions u defined in Ω such that

$$\|u\|_{L^p(\Omega)} = \Big(\int_\Omega |u(x)|^p dx\Big)^{1/p} < \infty.$$

We define the functional space $W_0^{1,p}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in the Sobolev space

$$W^{1,p}(\Omega) = \big\{u \in L^p(\Omega) \ : \ \nabla u \in L^p(\Omega)\big\},$$

with respect to the norm

$$\|u\|_{1,p} = \Big(\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega)}\Big)^{1/p}.$$

According to the Poincaré inequality, the norm $\|\cdot\|_{1,p}$ on $W_0^{1,p}(\Omega)$ is equivalent to the norm $\|\cdot\|_{W_0^{1,p}(\Omega)}$ setting by

$$\|u\|_{W_0^{1,p}(\Omega)} = \|\nabla u\|_{L^p(\Omega)} \ \text{ for } \ u \in W_0^{1,p}(\Omega).$$

Remember that the Sobolev space $W_0^{1,p}(\Omega)$ is a uniformly convex Banach space and the embedding $W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$ is compact (see Zeider (1990)).
Following that, we consider the functional space

$$\mathcal{V} := L^p(0,T;W_0^{1,p}(\Omega)), \quad (T>0)$$

that is a separable and reflexive Banach space with the norm

$$|u|_{\mathcal{V}} = \Big(\int_0^T \|u\|^p_{W_0^{1,p}(\Omega)} dt\Big)^{1/p}$$

or, by the Poincaré inequality, the equivalent norm in $\mathcal{V}$ given by

$$\|u\|_{\mathcal{V}} = \Big(\int_0^T \|\nabla u\|^p_{L^p(\Omega)} dt\Big)^{1/p}.$$

Next, we give some results and properties from the Berkovits and Mustonen degree theory for demicontinuous operators of generalized (S_+) type in real reflexive Banach. We start by defining some classes of mappings. In what follows, let X be a real separable reflexive Banach space with dual X^* and with continuous dual pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ and given a nonempty subset Ω of X, and $\rightharpoonup$ represents the weak convergence.
Let T from X to 2^{X^*} be a multi-values mapping. We denote by $Gr(T)$ the graph of T, i.e.

$$Gr(T) = \big\{(u,v) \in X \times X^* \ : \ v \in T(u)\big\}.$$

Definition 1. *The multi-values mapping T is called*

1. *monotone, if for each pair of elements $(\eta_1,\theta_1),(\eta_2,\theta_2)$ in $Gr(T)$, we have the inequality*

$$\langle \theta_1-\theta_2, \eta_1-\eta_2\rangle \geq 0.$$

2. *maximal monotone, if it is monotone and maximal in the sense of graph inclusion among monotone multi-values mappings from X to 2^{X^*}. An equivalent version of the last clause is that for any $(\eta_0,\theta_0) \in X \times X^*$ for which $\langle \theta_0-\theta, \eta_0-\eta\rangle \geq 0$, for all $(\eta,\theta) \in Gr(T)$, we have $(\eta_0,\theta_0) \in Gr(T)$.*

Let Y be another real Banach space.

Definition 2. *A mapping F: $D(F) \subset X \to Y$ is said to be*

1. *demicontinuous, if for each sequence $(u_n) \subset \Omega$, $u_n \to u$ implies $F(u_n) \rightharpoonup F(u)$.*
2. *of type (S_+), if for any sequence $(u_n) \subset D(F)$ with $u_n \rightharpoonup u$ and $\limsup\limits_{n\to\infty}\langle Fu_n, u_n-u\rangle \leq 0$, we have $u_n \to u$.*

Let $L \ : \ D(L) \subset X \to X^*$ be a linear maximal monotone map such that $D(L)$ is dense in X.
In the following, for each open and bounded subset G on X, we consider classes of operators :

$$\mathcal{F}_G(\Omega) := \big\{L+S : \overline{G} \cap D(L) \to X^* \mid S \ \text{ is bounded, demicontinuous and of type } (S_+) \text{ with respect to } D(L) \text{ from } G \text{ to } X^*\big\},$$

$$\mathcal{H}_G := \big\{L+S(t) \ : \ \overline{G} \cap D(L) \to X^* \mid S(t) \ \text{ is a bounded homotopy of type } (S_+) \text{ with respect to } D(L) \text{ from } \overline{G} \text{ to } X^*\big\}.$$

Remark 1. *Berkovits et al. (1992) Remark that the class $\mathcal{H}_G$ contains all affine homotopy*

$$L + (1-t)S_1 + tS_2 \quad \text{with} \quad (L + S_i) \in \mathcal{F}_G, \ \ i = 1, 2.$$

We give the Berkovits and Mustonen topological degree for a class of demicontinuous operator satisfying condition $(S_+)_T$ for more details see Berkovits et al. (1992).

Theorem 2. *Let L a linear maximal monotone densely defined map from $D(L) \subset X$ to X^* and*

$$M = \big\{(F, G, h) \ : \ F \in \mathcal{F}_G, \ G \text{ an open bounded subset in } X, \ h \notin F\big(\partial G \cap D(L)\big)\big\}.$$

There exists a unique degree function $d : M \longrightarrow \mathbb{Z}$ which satisfies the following properties :

1. *(Normalization) $L + J$ is a normalising map, where J is the duality mapping of X into X^*, that is, $d(L + J, G, h) = 1$, when $h \in (L + J)(G \cap D(L))$.*
2. *(Additivity) Let $F \in \mathcal{F}_G$. If G_1 and G_2 are two disjoint open subsets of G such that $h \notin F\big((\overline{G} \backslash (G_1 \cup G_2)) \cap D(L)\big)$ then we have*

$$d(F, G, h) = d(F, G_1, h) + d(F, G_2, h) \ .$$

3. *(Homotopy invariance) If $F(t) \in \mathcal{H}_G$ and $h(t) \notin F(t)(\partial G \cap D(L))$ for every $t \in [0, 1]$, where $h(t)$ is a continuous curve in X^*, then*

$$d(F(t), \ G, \ h(t)) = \ \text{constant}, \qquad \forall t \in [0, 1].$$

4. *(Existence) if $d(F, G, h) \neq 0$, then the equation $Fu = h$ has a solution in $G \cap D(L)$.*

Lemma 1. *Let $L + S \in \mathcal{F}_X$ and $h \in X^*$. Suppose that there exists $R > 0$ such that*

$$\langle Lu + Su - h, u\rangle > 0, \tag{5}$$

for any $u \in \partial B_R(0) \cap D(L)$. Then

$$(L + S)(D(L)) = X^*. \tag{6}$$

Proof. Let $\varepsilon > 0, \ t \in [0, 1]$ and

$$F_\varepsilon(t, u) = Lu + (1-t)Ju + t(Su + \varepsilon Ju - h).$$

As $0 \in L(0)$ and applying the boundary condition (5), we have

$$\begin{aligned}
\langle F_\varepsilon(t, u), u\rangle &= \langle t(Lu + Su - h, u\rangle + \langle (1-t)Lu + (1-t+\varepsilon)Ju, u\rangle \\
&\geq \langle (1-t)Lu + (1-t+\varepsilon)Ju, u\rangle \\
&= (1-t)\langle Lu, u\rangle + (1-t+\varepsilon)\langle Ju, u\rangle \\
&\geq (1-t+\varepsilon)\|u\|^2 = (1-t+\varepsilon)R^2 > 0.
\end{aligned}$$

Which means that $0 \notin F_\varepsilon(t, u)$. Since J and $S + \varepsilon J$ are bounded, continuous and of type (S_+), $\{F_\varepsilon(t, \cdot)\}_{t\in[0,1]}$ is an admissible homotopy. Hence, by using the normalisation and invariance under homotopy, we get

$$d(F_\varepsilon(t, \cdot), B_R(0), 0) = d(L + J, B_R(0), 0) = 1.$$

As a result, there exists $u_\varepsilon \in D(L)$ such that $0 \in F_\varepsilon(t, \cdot)$.
If we take $t = 1$ and when $\varepsilon \to 0^+$, then we have $h \in Lu + Su$ for some $u \in D(L)$. Since $h \in X^*$ is arbitrary, we deduce that $(L + S)(D(L)) = X^*$. □

4 Proof of Theorem 1

In this section, we give the proof of the main theorem. for that, we transform this nonlinear parabolic problem of Kirchhoff type (1) with Dirichlet boundary condition into a new one governed by operator equation of the form $Lu + Fu = g$. Then, using the theory of topological degrees introduced in the above section, we show the existence of weak solutions to the state problem. First, we give several lemmas.

Lemma 2. *Arosio & Panizzi (1996) Let $g \in L^r(Q)$ and $g_n \subset L^r(Q)$ such that $\|g_n\|_{r,\nu} \leq C$, $1 < r < \infty$, If $g_n(x) \to g(x)$ a.e. in Q then $g_n \rightharpoonup g$ weakly in $L^r(Q)$.*

Lemma 3. *Arosio & Panizzi (1996) Assume that (A_2)-(A_4) hold, let $(u_n)_n$ be a sequence in $\mathcal{V}$ such that $u_n \rightharpoonup u$ weakly in $\mathcal{V}$ and*

$$\int_Q \big[a(x,t,\nabla u_n) - a(x,t,\nabla u)\big]\nabla(u_n - u)dx \longrightarrow 0, \tag{7}$$

then $u_n \longrightarrow u$ strongly in $\mathcal{V}$.

Let us consider the following functional

$$\mathcal{E}(u) = \int_0^T \widehat{M}\Big(\int_\Omega A(x,t,\nabla u)dx\Big)dt, \qquad \text{for every } u \in \mathcal{V}$$

where $\widehat{M} : [0,+\infty[\longrightarrow [0,+\infty[$ be the primitive of the function M, defned by

$$\widehat{M}(s) = \int_0^s M(\xi)d\xi.$$

It is well known that $\mathcal{E}$ is well defined and continuously Gâteaux differentiable whose Gâteaux derivatives at point $u \in \mathcal{V}$ is the functional $\mathcal{E}'(u) \in \mathcal{V}^*$ setting by

$$\langle \mathcal{E}'(u)\,,\, v\rangle = \langle Fu\,,\, v\rangle, \qquad \text{for all } u, v \in \mathcal{V}$$

where the operator F acting from $\mathcal{V}$ to its dual $\mathcal{V}^*$ is defined by

$$\langle Fu\,,\, v\rangle = \int_0^T M\Big(\int_\Omega A(x,t,\nabla u)dx\Big)\int_\Omega a(x,t,\nabla u)\nabla v dxdt \tag{8}$$

for all $u, v \in \mathcal{V}$.

Proposition 1. *Suppose that (M_0),$(A_1) - (A_4)$ hold, then*

(i) F is bounded, strictly monotone, continuous operator.

(ii) F is of type (S_+).

Proof. i) It is clear that F is continuous, because F is the Fréchet derivative of $\mathcal{E}$.

First of all, let's prove that the operator F is bounded.

Let $u, v \in \mathcal{V}$, by the Hölder's inequality and (M_0), we obtain

$$\begin{aligned} |< Fu, v >| &= \Big|\int_0^T M\Big(\int_\Omega A(x,t,\nabla u)dx\Big)\int_\Omega a(x,t,\nabla u)\nabla v dxdt\Big| \\ &\leq \int_0^T m_1 \int_\Omega |a(x,t,\nabla u)\nabla v| dxdt \\ &\leq m_1 \int_0^T \|a(x,t,\nabla u)\|_{L^{p'}(\Omega)}\|\nabla v\|_{L^p(\Omega)}dt. \end{aligned}$$

From the growth condition (A_3), we can easily show that $\|a(x,t,\nabla u)\|_{L^{p'}(\Omega)}$ is bounded for all u in $W_0^{1,p}(\Omega)$. Therefore

$$|\langle Fu, v\rangle| \leq const \int_0^T \|\nabla v\|_{L^p(\Omega)} dt.$$

According to the continuous embedding $\mathcal{V} \hookrightarrow L^1(0,T,W_0^{1,p}(\Omega))$, we have

$$|\langle Fu, v\rangle| \leq const\, \|\nabla v\|_{\mathcal{V}},$$

as a result the operator F is bounded.

Next, we prove that F is strictly monotone operator.
For that, we consider the functional $B : W_0^{1,p}(\Omega) \to \mathbb{R}$ setting by

$$B(u) = \int_\Omega A(x,t,\nabla u)dx \quad \text{for all} \quad u \in W_0^{1,p}(\Omega),$$

so $B \in \mathcal{C}^1(W_0^{1,p}(\Omega,w),\mathbb{R})$ and

$$\langle B'(u), v\rangle = \int_\Omega a(x,t,\nabla u)\nabla v dx \qquad \text{for all } \ u,\, v \in W_0^{1,p}(\Omega).$$

By using (A_4), we obtain for any $u,\, v \in W_0^{1,p}(\Omega)$ with $u \neq v$

$$\langle B'(u) - B'(v), u - v\rangle > 0$$

which implies that B' is strictly monotone. Thus, by Prop. 25.10 in Zeider (1990), B is strictly convex. Furthermore, as M is nondecreasing, then $\widehat{M}$ is convex in $[0,+\infty[$. So, for any $u, v \in X$ with $u \neq v$, and every $s,\, r \in (0,1)$ with $s + r = 1$, we have

$$\widehat{M}(B(su + rv)) < \widehat{M}(sB(u) + rB(v)) \leq s\widehat{M}(B(u)) + r\widehat{M}(B(v)).$$

This proves that $\mathcal{E}$ is strictly convex, since $\mathcal{E}'(u) = F(u)$ in $\mathcal{V}^*$, then we infer that F is strictly monotone in $\mathcal{V}$.

$ii)-$ It remains to prove that the operator F is of type (S_+).
Let $(u_n)_n$ be a sequence in $D(F)$ such that

$$\begin{cases} u_n \rightharpoonup u \qquad \text{in } \mathcal{V} \\ \limsup\limits_{n\to\infty} \langle Fu_n, u_n - u\rangle \leq 0. \end{cases}$$

We will show that $u_n \to u$ in $\mathcal{V}$.

On the one hand, in fact $u_n \rightharpoonup u$ in $\mathcal{V}$, so $(u_n)_n$ is a bounded sequence in $\mathcal{V}$, then there exist a subsequence still denoted by $(u_n)_n$ such that $u_n \rightharpoonup u$ in $\mathcal{V}$, under the strict monotonicity of F we get

$$0 = \limsup_{n\to\infty} \langle Fu_n - Fu,\ u_n - u\rangle = \lim_{n\to\infty} \langle Fu_n - Fu,\ u_n - u\rangle.$$

Then

$$\lim_{n\to\infty} \langle Fu_n,\ u_n - u\rangle = 0 \quad \text{and} \quad \lim_{n\to\infty} \langle Fu,\ u_n - u\rangle = 0,$$

which means

$$\lim_{n\to\infty} \int_0^T M\Big(\int_\Omega A(x,t,\nabla u_n)dx\Big) \int_\Omega a(x,t,\nabla u_n)\nabla(u_n - u)dxdt = 0, \tag{9}$$

and

$$\lim_{n\to\infty}\int_0^T M\Big(\int_\Omega A(x,t,\nabla u)dx\Big)\int_\Omega a(x,t,\nabla u)\nabla(u_n-u)dxdt=0. \tag{10}$$

On the one hand, according to (M_0) we obtain

$$\begin{aligned} m_0\int_Q a(x,t,\nabla u)\nabla(u_n-u)dxdt \\ \leq \int_0^T M\Big(\int_\Omega A(x,t,\nabla u)dx\Big)\int_\Omega a(x,t,\nabla u)\nabla(u_n-u)dxdt \\ \leq m_1\int_Q a(x,t,\nabla u)\nabla(u_n-u)dxdt. \end{aligned} \tag{11}$$

By combining (10) and (11) we deduce that

$$\lim_{n\to\infty}\int_Q a(x,t,\nabla u)\nabla(u_n-u)dxdt=0. \tag{12}$$

On the other hand, by (A_1) we have for any $x\in\Omega$ and $\xi\in\mathbb{R}^n$

$$A(x,t,\xi)=\int_0^1\frac{d}{ds}A(x,t,s\xi)ds=\int_0^1 a(x,t,s\xi)\xi ds$$

By combining (A_3), Fubini's theorem and Hölder's inequality we have

$$\begin{aligned} \int_\Omega A(x,t,\nabla u_n)dx &= \int_\Omega\int_0^1 a(x,t,s\nabla u_n)\nabla u_n dsdx \\ &= \int_0^1\int_\Omega a(x,t,s\nabla u_n)\nabla u_n dxds \\ &\leq \int_0^1\beta\int_\Omega\big(d(x,t)|\nabla u_n|+|s\nabla u_n|^p\big)dxds \\ &\leq \beta\int_\Omega|d(x,t)||\nabla u_n|dx+\int_0^1\int_\Omega s^p|\nabla u_n|^p\big)dxds \\ &\leq C_1\|u_n\|^p_{W_0^{1,p}(\Omega)}+C'\int_\Omega|\nabla u_n|^p dx \\ &\leq C\|u_n\|^p_{W_0^{1,p}(\Omega)}. \end{aligned}$$

Then, we infer that $\Big(\int_\Omega(A(x,t,\nabla u_n)dx\Big)_{n\geq 1}$ is bounded.
As M is continuous, up to a subsequence there is $s_0\geq 0$ such that

$$M\Big(\int_\Omega(A(x,t,\nabla u_n)dx\Big)\longrightarrow M(s_0)\geq m_0 \qquad \text{as} \quad n\to\infty. \tag{13}$$

In the view of (9), (13), (A_2), (A_3) and the Lebesgue dominated convergence theorem, we get

$$\lim_{n\to\infty}\int_0^T\int_\Omega a(x,t,\nabla u_n)\nabla(u_n-u)dxdt=0. \tag{14}$$

From (12) and (14), we have

$$\lim_{n\to\infty}\int_Q\big[a(x,t,\nabla u_n)-a(x,t,\nabla u)\big]\nabla(u_n-u)dx=0.$$

In light of Lemma 3, we obtain

$$u_n\longrightarrow u \qquad \text{strongly in } \mathcal{V},$$

which implies that F is of type (S_+).

□

Proof of Theorem 1.

Let us consider the following operator L defined from $\mathcal{V} \supset D(L)$ into its dual $\mathcal{V}^*$, such that

$$D(L) = \big\{v \in \mathcal{V} \ : \ v' \in \mathcal{V}^*, \ v(0) = 0\big\},$$

by

$$\langle Lu, v\rangle = -\int_Q uvdxdt, \qquad \text{for all } u \in D(L), \ v \in \mathcal{V}.$$

Consequently, the operator L is generated by $\partial/\partial t$ by means of the relation

$$\langle Lu, v\rangle = \int_0^T \langle u'(t), v(t)\rangle dt, \ \text{ for all } u \in D(L), \ v \in \mathcal{V}.$$

We can confirm, like in Zeider (1990) that L is a densely defined maximal monotone operator. Combining the condition of connectivity (A_4) and the monotonicity of L $\big(\langle Lu, u\rangle \geq 0$ for all $u \in D(L)\big)$, we obtain

$$\begin{aligned}
\langle Lu + Fu, u\rangle &\geq \langle Fu, u\rangle \\
&= \int_0^T M\Big(\int_\Omega A(x,t,\nabla u)dx\Big)\int_\Omega a(x,t,\nabla u)\nabla u dxdt \\
&\geq \int_0^T m_0 \int_\Omega a(x,t,\nabla u)\nabla u dxdt \\
&\geq m_0 \int_Q a(x,t,\nabla u)\nabla u dxdt \qquad (15)\\
&\geq m_0 \int_Q |\nabla u|^p dxdt \\
&= C'\|u\|_{\mathcal{V}}^p
\end{aligned}$$

for all $u \in \mathcal{V}$.

Since the right hand side in (15) tends to ∞ as $\|u\|_{\mathcal{V}} \to \infty$, so for every $g \in \mathcal{V}^*$ there exists $R = R(g)$ for which $\langle Lu + Fu - g, u\rangle > 0$ for all $u \in B_R(0) \cap D(L)$.

By using the Lemma 1, we infer that the equation $Lu + Fu = g$ is solvable in $D(L)$.

Which implies that the problem (1) admits at least one weak solution. This ends the proof. □

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